APPENDIX

A. Properties of the Value-at-Risk

Lemma 2. (Properties of the Value-at-Risk). *Given a random variable $X \sim \mathbb{P} = \mathcal{N}(\mu, \sigma^2)$ and a threshold $\epsilon \in [0, 1)$, the following holds.*

(a) $VaR_\epsilon^{\mathbb{P}}[-|X|^2] = -VaR_\epsilon^{\mathbb{P}}[-|X|]^2$

(b) For $Y = -|X|$ with a fixed σ^2,

$$\frac{dVaR_\epsilon^{\mathbb{P}}[Y]}{d\mu} = \frac{\phi(\frac{VaR_\epsilon^{\mathbb{P}}[Y]-\mu}{\sigma}) - \phi(\frac{-VaR_\epsilon^{\mathbb{P}}[Y]-\mu}{\sigma})}{\phi(\frac{VaR_\epsilon^{\mathbb{P}}[Y]-\mu}{\sigma}) + \phi(\frac{-VaR_\epsilon^{\mathbb{P}}[Y]-\mu}{\sigma})},$$

where ϕ is the probability density function of the standard normal distribution.

(c) $VaR_\epsilon^{\mathbb{P}}[-|X|] = \sqrt{2}\sigma \cdot erf^{-1}(\epsilon - 1)$

(d) $VaR_\epsilon^{\mathbb{P}}[X] > VaR_\epsilon^{\mathbb{P}}[-|X|]$

Proof: (a)

$$\begin{aligned} &P[-|X| \le \text{VaR}_\epsilon^{\mathbb{P}}[-|X|]] = \epsilon, \\ &P[-|X|^2 \le \text{VaR}_\epsilon^{\mathbb{P}}[-|X|^2]] = \epsilon \\ \Leftrightarrow &P[|X| \ge \sqrt{-\text{VaR}_\epsilon^{\mathbb{P}}[-|X|^2]}] = \epsilon \\ \Leftrightarrow &P[-|X| \le -\sqrt{-\text{VaR}_\epsilon^{\mathbb{P}}[-|X|^2]}] = \epsilon. \\ &\therefore \text{VaR}_\epsilon^{\mathbb{P}}[-|X|^2] = -\text{VaR}_\epsilon^{\mathbb{P}}[-|X|]^2 \end{aligned}$$

(b) Let $F_Y(y)$ and $f_Y(y)$ be the cumulative probability distribution and the probability density function of a random variable Y, respectively. Let us similarly define $F_X(x)$ and $f_X(x)$ for a random variable X. Also, let Φ be the cumulative distribution function of the standard normal distribution. For simplicity, we denote $\text{Var}_\epsilon^{\mathbb{P}}[Y]$ by $k(\mu)$, which is also a function of σ but we consider it as a constant.

By definition of VaR, we have

$$\begin{aligned} \epsilon = F_Y(k(\mu)) &= \int_{-\infty}^{k(\mu)} f_Y(y)dy \\ &= \int_{-\infty}^{k(\mu)} (f_X(x) + f_X(-x))dx \\ &= \int_{-\infty}^{k(\mu)} f_X(x)dx + \int_{-\infty}^{k(\mu)} f_X(-x)dx. \end{aligned}$$

Let $t = -x$. Then,

$$\begin{aligned} F_Y(k(\mu)) &= \int_{-\infty}^{k(\mu)} f_X(x)dx + \int_{\infty}^{-k(\mu)} f_X(t)(-dt) \\ &= \int_{-\infty}^{k(\mu)} f_X(x)dx - \int_{\infty}^{-k(\mu)} f_X(x)dx \\ &= \Phi\left(\frac{k(\mu)-\mu}{\sigma}\right) - \Phi\left(\frac{-k(\mu)-\mu}{\sigma}\right) + 1. \end{aligned}$$

Because $\epsilon = F_Y(k(\mu))$, we have $\frac{dF_Y(k(\mu))}{d\mu} = 0$. This leads to

$$\phi\left(\frac{k(\mu)-\mu}{\sigma}\right) \cdot \left(\frac{dk(\mu)}{d\mu} - 1\right)\frac{1}{\sigma} - \phi\left(\frac{-k(\mu)-\mu}{\sigma}\right) \cdot \left(-\frac{dk(\mu)}{d\mu} - 1\right)\frac{1}{\sigma} = 0.$$

Then

$$\frac{dk(\mu)}{d\mu} = \frac{\phi\left(\frac{k(\mu)-\mu}{\sigma}\right) - \phi\left(\frac{-k(\mu)-\mu}{\sigma}\right)}{\phi\left(\frac{k(\mu)-\mu}{\sigma}\right) + \phi\left(\frac{-k(\mu)-\mu}{\sigma}\right)}.$$

This concludes the proof.

(c) Let $-|X| = Y$. Then the probability density function of Y is

$$f_Y(y) = f_X(x) + f_X(-x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} exp(-\frac{y^2}{2\sigma^2}), y \le 0.$$

And the cumulative distribution function of Y is

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{y} \frac{\sqrt{2}}{\sigma\sqrt{\pi}} exp(-\frac{t^2}{2\sigma^2})dt.$$

Meanwhile,

$$F_Y(0) = \int_{-\infty}^{0} f_Y(t)dt = 1$$

because Y is defined in $(-\infty, 0]$. Then

$$\begin{aligned} F_Y(0) &= \int_{-\infty}^{y} f_Y(t)dt + \int_{y}^{0} f_Y(t)dt \\ &= \int_{-\infty}^{y} f_Y(t)dt - \int_{0}^{y} f_Y(t)dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{y} f_Y(t)dt &= 1 + \int_{0}^{y} f_Y(t)dt \\ &= 1 + \int_{0}^{y} \frac{\sqrt{2}}{\sigma\sqrt{\pi}} exp(-\frac{t^2}{2\sigma^2})dt. \end{aligned}$$

Let $t/\sqrt{2}\sigma = k$. Then

$$\begin{aligned} F_Y(y) &= 1 + \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \int_{0}^{y/\sqrt{2}\sigma} exp(-k^2) \cdot \sqrt{2}\sigma dk \\ &= 1 + \frac{2}{\sqrt{\pi}} \int_{0}^{y/\sqrt{2}\sigma} exp(-k^2)dk \\ &= 1 + erf(y/\sqrt{2}\sigma). \end{aligned}$$

$\text{VaR}_\epsilon^{\mathbb{P}}[Y]$ is the value of y when $F_Y(y) = \epsilon$. Therefore,

$$\text{VaR}_\epsilon^{\mathbb{P}}[Y] = \sqrt{2}\sigma \cdot erf^{-1}(\epsilon - 1).$$

(d) Let $\text{VaR}_\epsilon^{\mathbb{P}}[X] = k, \text{VaR}_\epsilon^{\mathbb{P}}[-|X|] = k'$. By definition,

$$\int_{k}^{\infty} f_X(x)dx = 1 - \epsilon = \int_{k'}^{0} f_{-|X|}(x)dx.$$

Also,

$$\begin{aligned} &\int_{k}^{\infty} f_X(x)dx \\ &= \int_{k}^{0} f_X(x)dx + \int_{0}^{-k} f_X(x)dx + \int_{-k}^{\infty} f_X(x)dx \\ &= \int_{k}^{0} f_X(x)dx + \int_{k}^{0} f_X(-x)dx + \int_{-k}^{\infty} f_X(x)dx \\ &= \int_{k}^{0} f_{-|X|}(x)dx + \int_{-k}^{\infty} f_X(x)dx. \end{aligned}$$

Therefore,

$$\int_{-k}^{\infty} f_X(x)dx > 0$$
$$\implies \int_{k'}^{0} f_{-|X|}(x)dx - \int_{k}^{0} f_{-|X|}(x)dx$$
$$= \int_{k'}^{k} f_{-|X|}(x)dx > 0.$$

Therefore, $\text{VaR}_\epsilon^{\mathbb{P}}[X] > \text{VaR}_\epsilon^{\mathbb{P}}[-|X|]$.

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B. Properties of the Conditional Value-at-Risk

Lemma 1. (Properties of the Conditional Value-at-Risk). *Given a random variable $X \sim \mathbb{P} = \mathcal{N}(\mu, \sigma^2)$ and a threshold $\epsilon \in [0, 1)$, the following holds.*

(a) $CVaR_\epsilon^{\mathbb{P}}[-|X|^2] \leq -CVaR_\epsilon^{\mathbb{P}}[-|X|]^2$

(b) For a fixed σ^2, $CVaR_\epsilon^{\mathbb{P}}[-|X|]$ is monotonically decreasing with respect to $|\mu|$.

(c) If $\mu = 0$, then $CVaR_\epsilon^{\mathbb{P}}[-|X|] = \kappa \cdot \sigma$, where $\kappa = \frac{1}{1-\epsilon}\sqrt{2/\pi}[exp(-[erf^{-1}(\epsilon - 1)]^2) - 1]$.

(d) If $VaR_\epsilon^{\mathbb{P}}[X] > 0 \wedge \mu > 0$, or $VaR_\epsilon^{\mathbb{P}}[X] < 0 \wedge \mu < 0$, then $-|\mu| + \delta \cdot \sigma > CVaR_\epsilon^{\mathbb{P}}[-|X|]$.

Proof:

(a)

$$\text{CVaR}_\epsilon^{\mathbb{P}}[-|X|^2] = \mathbb{E}[-|X|^2 : -|X|^2 \geq \text{VaR}_\epsilon^{\mathbb{P}}[-|X|^2]] \quad (14)$$
$$= \mathbb{E}[-|X|^2 : -|X|^2 \geq -\text{VaR}_\epsilon^{\mathbb{P}}[-|X|]^2] \quad (15)$$
$$= \mathbb{E}[-|X|^2 : |X| \leq -\text{VaR}_\epsilon^{\mathbb{P}}[-|X|]] \quad (16)$$
$$= -\mathbb{E}[|X|^2 : |X| \leq -\text{VaR}_\epsilon^{\mathbb{P}}[-|X|]] \quad (17)$$
$$\leq -\mathbb{E}[-|X| : |X| \leq -\text{VaR}_\epsilon^{\mathbb{P}}[-|X|]]^2 \quad (18)$$
$$= -\mathbb{E}[-|X| : -|X| \geq \text{VaR}_\epsilon^{\mathbb{P}}[-|X|]]^2 \quad (19)$$
$$= -\text{CVaR}_\epsilon^{\mathbb{P}}[-|X|]^2. \quad (20)$$

Equation (15) holds by Lemma 2 (a), and inequality (18) holds by Jensen's inequality.

(b) With the same notations used in the proof of Lemma 2 (b), we have

$$\text{CVaR}_\epsilon^{\mathbb{P}}[Y] = \mathbb{E}[Y : Y \geq k(\mu)] = \frac{1}{1-\epsilon}\int_{k(\mu)}^{0} y f_Y(y)dy$$
$$= \frac{1}{1-\epsilon}\int_{k(\mu)}^{0} x(f_X(x) + f_X(-x))dx$$
$$= \frac{1}{1-\epsilon}\left[\int_{k(\mu)}^{0} x f_X(x)dx + \int_{-k(\mu)}^{0} t f_X(t)dt\right]$$
$$= \frac{1}{1-\epsilon}\left[\int_{k(\mu)}^{0} x f_X(x)dx - \int_{0}^{-k(\mu)} x f_X(x)dx\right].$$

Let $a(\mu), b(\mu)$ be functions with respect to μ, and standardized random variable $Z = (X - \mu)/\sigma$. Then,

$$\int_{a(\mu)}^{b(\mu)} x f_X(x)dx = \int_{\frac{a(\mu)-\mu}{\sigma}}^{\frac{b(\mu)-\mu}{\sigma}} (\mu + \sigma \cdot z)\frac{1}{\sigma}\phi(z)\sigma dz$$
$$= \mu\int_{\frac{a(\mu)-\mu}{\sigma}}^{\frac{b(\mu)-\mu}{\sigma}} \phi(z)dz + \sigma\int_{\frac{a(\mu)-\mu}{\sigma}}^{\frac{b(\mu)-\mu}{\sigma}} z\phi(z)dz.$$

With $z_a(\mu) = \frac{a(\mu)-\mu}{\sigma}$ and $z_b(\mu) = \frac{b(\mu)-\mu}{\sigma}$, the derivative with respect to μ is

$$\frac{d}{d\mu}\int_{a(\mu)}^{b(\mu)} x f_X(x)dx =$$
$$\left[\int_{z_a(\mu)}^{z_b(\mu)} \phi(z)dz + \mu \cdot \{\phi(z_b(\mu)) \cdot z_b'(\mu) - \phi(z_a(\mu)) \cdot z_a'(\mu)\}\right]$$
$$+ \sigma\left[z_b(\mu)\phi(z_b(\mu))z_b'(\mu) - z_a(\mu)\phi(z_a(\mu))z_a'(\mu)\right]$$
$$= \int_{z_a(\mu)}^{z_b(\mu)} \phi(z)dz + b(\mu)\phi(z_b(\mu))z_b'(\mu) - a(\mu)\phi(z_a(\mu))z_a'(\mu).$$

Using this result and letting $z_k(\mu) = \frac{k(\mu)-\mu}{\sigma}, z_{-k}(\mu) = \frac{-k(\mu)-\mu}{\sigma}$ and $z_0(\mu) = \frac{-\mu}{\sigma}$, we can express the derivative of $\text{CVaR}_\epsilon^{\mathbb{P}}[Y]$ with respect to μ as follows.

$$(1-\epsilon)\frac{d}{d\mu}\text{CVaR}_\epsilon^{\mathbb{P}}[Y]$$
$$= \frac{d}{d\mu}\left[\int_{k(\mu)}^{0} x f_X(x)dx - \int_{0}^{-k(\mu)} x f_X(x)dx\right]$$
$$= \int_{z_k(\mu)}^{z_0(\mu)} \phi(z)dz - k(\mu) \cdot \phi(z_k(\mu)) \cdot z_k'(\mu)$$
$$- \int_{z_0(\mu)}^{z_{-k}(\mu)} \phi(z)dz - \left[(-k(\mu)) \cdot \phi(z_{-k}(\mu)) \cdot z_{-k}'(\mu)\right]$$
$$= \int_{z_k(\mu)}^{z_0(\mu)} \phi(z)dz - \int_{z_0(\mu)}^{z_{-k}(\mu)} \phi(z)dz$$
$$- k(\mu)\left[\phi(z_k(\mu)) \cdot z_k'(\mu) - \phi(z_{-k}(\mu)) \cdot z_{-k}'(\mu)\right].$$

Because

$$z_k'(\mu) = \frac{k'(\mu) - 1}{\sigma}, \qquad z_{-k}'(\mu) = \frac{-k'(\mu) - 1}{\sigma},$$

and by Lemma 2 (b),

$$k'(\mu) = \frac{dk(\mu)}{d\mu} = \frac{\phi(z_k(\mu)) - \phi(z_{-k}(\mu))}{\phi(z_k(\mu)) + \phi(z_{-k}(\mu))},$$

we have $\phi(z_k(\mu)) \cdot z_k'(\mu) - \phi(z_{-k}(\mu)) \cdot z_{-k}'(\mu) = 0$. Therefore,

$$\frac{d}{d\mu}\text{CVaR}_\epsilon^{\mathbb{P}}[Y] = \frac{1}{1-\epsilon}\left[\int_{\frac{k(\mu)-\mu}{\sigma}}^{\frac{-\mu}{\sigma}} \phi(z)dz - \int_{\frac{-\mu}{\sigma}}^{\frac{-k(\mu)-\mu}{\sigma}} \phi(z)dz\right].$$

Note that for any $b \leq 0$, $\int_{a+b}^{a} \phi(z)dz > \int_{a}^{a-b} \phi(z)dz$ if $a > 0$ and $\int_{a+b}^{a} \phi(z)dz < \int_{a}^{a-b} \phi(z)dz$ if $a < 0$. Therefore,

$$\frac{d}{d\mu}\text{CVaR}_\epsilon^{\mathbb{P}}[Y] < 0 \text{ for } \mu > 0.$$

and

$$\frac{d}{d\mu}\text{CVaR}_\epsilon^{\mathbb{P}}[Y] > 0 \text{ for } \mu < 0.$$

That is, $\text{CVaR}_\epsilon^{\mathbb{P}}[Y]$ is monotonically decreasing with respect to $|\mu|$.

(c) Let $-|X| = Y$ and $\text{VaR}_\epsilon[Y] = k$. Then

$$\begin{aligned}\text{CVaR}_\epsilon^{\mathbb{P}}[Y] &= \mathbb{E}[Y : Y \geq k] \\ &= \frac{1}{1-\epsilon}\int_k^0 y f_Y(y) dy \\ &= \frac{1}{1-\epsilon}\int_k^0 y \cdot \frac{\sqrt{2}}{\sqrt{\pi}\sigma} exp(-\frac{y^2}{2\sigma^2}) dy.\end{aligned}$$

Let $y/\sqrt{2}\sigma = u, y = \sqrt{2}\sigma u, dy = \sqrt{2}\sigma du$. Then

$$\begin{aligned}&= \frac{1}{1-\epsilon}\int_{k/\sqrt{2}\sigma}^0 \sqrt{2}\sigma u \cdot \frac{\sqrt{2}}{\sqrt{\pi}\sigma} exp(-u^2)\sqrt{2}\sigma du \\ &= \frac{1}{1-\epsilon} \cdot \frac{2\sqrt{2}}{\sqrt{\pi}}\sigma \int_{k/\sqrt{2}\sigma}^0 u \cdot exp(-u^2) du \\ &= \frac{1}{1-\epsilon} \cdot \frac{2\sqrt{2}}{\sqrt{\pi}}\sigma \left[-\frac{1}{2} exp(-u^2)\right]_{k/\sqrt{2}\sigma}^0 \\ &= \frac{1}{1-\epsilon} \cdot \frac{\sqrt{2}}{\sqrt{\pi}}\sigma \left[exp(-\left[k/\sqrt{2}\sigma\right]^2) - 1\right] \\ &= \frac{1}{1-\epsilon} \cdot \frac{\sqrt{2}}{\sqrt{\pi}}\sigma \left[exp(-\left[\frac{\sqrt{2}\sigma \cdot erf^{-1}(\epsilon-1)}{\sqrt{2}\sigma}\right]^2) - 1\right] \\ &= \frac{1}{1-\epsilon} \cdot \frac{\sqrt{2}}{\sqrt{\pi}}\sigma \left[exp(-[erf^{-1}(\epsilon-1)]^2) - 1\right] \\ &= \kappa \cdot \sigma,\end{aligned}$$

where $k = \sqrt{2}\sigma \cdot erf^{-1}(\epsilon - 1)$ by Lemma 2 (c) and $\kappa = \frac{1}{1-\epsilon}\sqrt{2/\pi}\left[exp(-[erf^{-1}(\epsilon-1)]^2) - 1\right]$.

(d) If $\mu < 0$ and $\text{VaR}_\epsilon^{\mathbb{P}}[X] = k < 0$, then

$$\begin{aligned}&\text{CVaR}_\epsilon^{\mathbb{P}}[X] = \mathbb{E}[X|X \geq k] \\ &= \frac{1}{1-\epsilon}\int_k^\infty x f_X(x) dx \\ &> \frac{1}{1-\epsilon}\int_k^{-k} x f_X(x) dx \\ &= \frac{1}{1-\epsilon}[\int_k^0 x f_X(x) dx + \int_0^{-k} x f_X(x) dx] \\ &= \frac{1}{1-\epsilon}[\int_k^0 x f_X(x) dx + \int_0^{k} x f_X(-x) dx] \\ &= \frac{1}{1-\epsilon}[\int_k^0 x f_X(x) dt - \int_k^{0} x f_X(-x) dx] \\ &\geq \frac{1}{1-\epsilon}[\int_k^0 x f_X(x) dt + \int_k^{0} x f_X(-x) dx] \\ &= \frac{1}{1-\epsilon}\int_k^0 x(f_X(-x) + f_X(x)) dx \\ &= \frac{1}{1-\epsilon}\int_k^0 x f_{-|X|}(x) dx \\ &\geq \frac{1}{1-\epsilon}\int_{\text{VaR}_\epsilon^{\mathbb{P}}[-|X|]}^0 x f_{-|X|}(x) dx \\ &= \mathbb{E}[-|X| : -|X| \geq \text{VaR}_\epsilon^{\mathbb{P}}[-|X|]] \\ &= \text{CVaR}_\epsilon^{\mathbb{P}}[-|X|].\end{aligned}$$

The last inequality is because $k > \text{VaR}_\epsilon^{\mathbb{P}}[-|X|]$ by Lemma 2 (d). Therefore, $\text{CVaR}_\epsilon^{\mathbb{P}}[X] = \mu + \delta \cdot \sigma > \text{CVaR}_\epsilon^{\mathbb{P}}[-|X|]$.

Similarly, if $\mu > 0$ and $\text{VaR}_\epsilon^{\mathbb{P}}[X] > 0$, the mean of $-X$ is smaller than zero and $\text{VaR}_\epsilon^{\mathbb{P}}[-X] < 0$. Thus, we can use the inequality that we derived in the preceding paragraph, that is,

$$\text{CVaR}_\epsilon^{\mathbb{P}}[-X] > \text{CVaR}_\epsilon^{\mathbb{P}}[-|X|].$$

Because X has a symmetric distribution about the mean μ, we have $-\text{VaR}_\epsilon^{\mathbb{P}}[-X] = \text{VaR}_{1-\epsilon}^{\mathbb{P}}[X]$. Also, $\text{CVaR}_{1-\epsilon}^{\mathbb{P}}[X] = \mathbb{E}[X : X \geq \text{VaR}_\epsilon^{\mathbb{P}}[X]] = \mu - \delta \cdot \sigma$ by [29]. Thus,

$$\begin{aligned}\text{CVaR}_\epsilon^{\mathbb{P}}[-X] &= \mathbb{E}[-X : -X \geq \text{VaR}_\epsilon^{\mathbb{P}}[-X]] \\ &= -\mathbb{E}[X : X \leq -\text{VaR}_\epsilon^{\mathbb{P}}[-X]] \\ &= -\mathbb{E}[X : X \leq \text{VaR}_{1-\epsilon}^{\mathbb{P}}[X]] \\ &= -(\mu - \delta \cdot \sigma).\end{aligned}$$

Therefore, if $\text{VaR}_\epsilon^{\mathbb{P}}[X] > 0 \wedge \mu > 0$ or $\text{VaR}_\epsilon^{\mathbb{P}}[X] < 0 \wedge \mu < 0$, then $-|\mu| + \delta \cdot \sigma > \text{CVaR}_\epsilon^{\mathbb{P}}[-|X|]$. ■